The Conformable Kalman Filter

Nathan Murarik [1] Gino Rotellini [2] Tatiana Sosnovsky [3] Nick Wintz [4]

[1]Department of Mathematics - Pennsylvania State University (nkm5423@psu.edu)
[2]Department of Mathematics - Pennsylvania Western University (rot9862@pennwest.edu)
[3]Department of Mathematics and Statistics- California State Polytechnic University, Pomona (tsosnovsky@cpp.edu)
[4]Department of Mathematics, Computer Science, and Information Technology - Lindenwood University (nwintz@lindenwood.edu)

Abstract

In this project, we formulate the Kalman-Bucy filter for linear, continuous conformable control systems corrupted by white noise. Here, we use the system first introduced by Khalil, et al. We obtain a state transition matrix via a Peano-Baker expansion that allows us to calculate our error propagation through a Riccati equation. In addition, we show the duality between the conformable Kalman filter and its associated conformable linear quadratic regulator (CLQR) problem is preserved. Finally, we provide numerical simulations for relevant applications.

Definition (Conformable Derivative)

Let $f:[0,\infty)\to\mathbb{R}$ and let $\alpha\in(0,1]$. Then the conformable derivative of order α of f at t is defined by

$$f^{\alpha}(t):=\begin{cases}\lim\limits_{\theta\to 0}\dfrac{f(t+\theta t^{1-\alpha})-f(t)}{\theta}, & t>0\\ \lim\limits_{s\to 0^+} f^{(\alpha)}(s), & t=0,\end{cases}$$

provided that the limit exists.

Properties

Let $\alpha\in(0,1]$, f,g be $\alpha-$differentiable functions for $t>0$, and $a,b\in\mathbb{R}$. Then

- $(af+bg)^{(\alpha)}(t)=af(t)^{(\alpha)}+bg(t)^{(\alpha)}$,
- $(t^b)^{(\alpha)}=bt^{b-\alpha}$,
- $(b)^{(\alpha)}=0$,
- $(fg)^{(\alpha)}(t)=f(t)^{(\alpha)}g(t)+f(t)g(t)^{(\alpha)}$,
- $\left(\frac{f}{g}\right)^{(\alpha)}(t)=\frac{g(t)f^{(\alpha)}(t)-f(t)g^{(\alpha)}(t)}{[g(t)]^2}$, and
- if f is differentiable, then $f^{(\alpha)}(t)=t^{1-\alpha}f'(t)$.

Definition (Conformable Integral)

$I^{\alpha}_{a}(f)(t)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann integral and $\alpha\in(0,1]$.

Peano-Baker Series for Conformable Systems

The state transition matrix is given by

$$\Phi_A(t,t_0)=I+\int_{t_0}^{t}\frac{A(\tau_1)}{\tau_1^{1-\alpha}}d\tau_1+\int_{t_0}^{t}\frac{A(\tau_1)}{\tau_1^{1-\alpha}}\int_{t_0}^{\tau_1}\frac{A(\tau_2)}{\tau_2^{1-\alpha}}d\tau_2d\tau_1+\cdots$$
$$+\int_{t_0}^{t}\frac{A(\tau_1)}{\tau_1^{1-\alpha}}\int_{t_0}^{\tau_1}\frac{A(\tau_2)}{\tau_2^{1-\alpha}}\cdots\int_{t_0}^{\tau_{k-1}}\frac{A(\tau_k)}{\tau_k^{1-\alpha}}d\tau_kd\tau_{k-1}\cdots d\tau_1+\cdots,$$

where $\Phi(\cdot,t_0)$ satisfies

$$x^{(\alpha)}(t)=A(t)x(t),\ x(t_0)=x_0.$$

Properties of Peano-Baker Expansion

- $\Phi_A(t,t_0)\Phi_A(t_0,t_1)=\Phi_A(t,t_1)$
- $\Phi_A(t,t)=I$
- $\Phi(\cdot,\cdot)$ is invertible
- $\Phi_A^{(\alpha)}(t,t_0)=A(t)\Phi_A(t,t_0)$

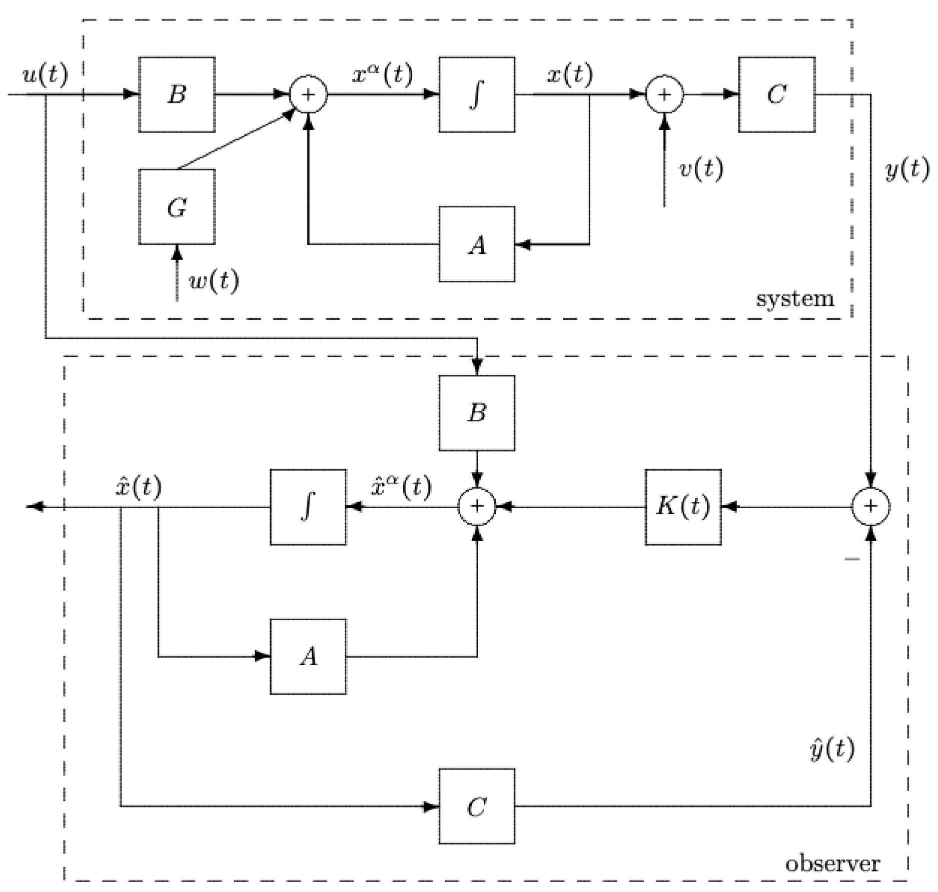
Stochastic System

Consider the system

$$x^{(\alpha)}(t)=A(t)x(t)+B(t)u(t)+Gw(t),\ x(t_0)=x_0$$
$$y(t)=C(t)x(t)+v(t)$$

where

- $x\in\mathbb{R}^n$ is the state,
- $u\in\mathbb{R}^m$ is the deterministic control,
- $y\in\mathbb{R}^p$ is the measurement,
- $w\in\mathbb{R}^l$ is the process noise, and
- $v\in\mathbb{R}^p$ is the measurement noise.



The Conformable Kalman Filter (CKF)

System: $x^{(\alpha)}(t)=A(t)x(t)+B(t)u(t)+Gw(t)$
Measurement: $y(t)=C(t)x(t)+v(t)$

Assumptions: $x_0\sim(\bar{x}_0,P_0)$, $w\sim(0,Q\delta(t-s))$, $v\sim(0,R\delta(t-s))$, which are mutually uncorrelated, R > 0

Initialization
Initial Estimate: $\hat{x}(t_0)=\bar{x}_0$
Error Covariance: $P(t_0)=\mathbb{E}[(x_0-\hat{x}_0)(x_0-\hat{x}_0)^T]=P_0$

Estimate Update:
$\hat{x}^{(\alpha)}(t)=A(t)\hat{x}(t)+B(t)u(t)+K(t)[y(t)-C(t)\hat{x}(t)]$

Kalman Gain: $K(t)=P(t)C^T(t)(R^{-1})$

Error Covariance Update:
$P^{(\alpha)}(t)=A(t)P(t)+P(t)A^T(t)-P(t)C(t)^TR^{-1}C(t)P(t)+GQG^T$

CLQR vs CKF

When comparing Riccati equations,

$$CLQR:-S^{\alpha}=A^TS+SA-SBW^{-1}B^TS+Q$$
$$CKF:P^{\alpha}=AP+PA^T-PC^TR^{-1}CP+GQG^T,$$

we see the duality property has been preserved. Below is a comparison of matrix weights.

CLQR	CKF
A^T	A
S	P
B	C^T
W^{-1}	R^{-1}
Q	GQG^T

By the separability principle, we obtain

$$\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}^{\alpha}(t)=\begin{bmatrix}A-BL(t) & BL(t)\\ 0 & A-K(t)C\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}(t),$$

where K and L are the gains of the CKF and CLQR respectively.

CKF Algorithm

```
procedure CKF( n ∈ ℕ, dt ∈ (0,∞), α ∈ (0,1] )
    Objective Function: x̂_i ≈ x_i such that P_i is bounded
    Initialize P_n, x_n, x̂_n, w_n, and v_n as arrays of length n
    Initialize P_1, x_1, x̂_1, w_1, and v_1
    Initialize A, B, C, G, Q, and R
    factor ← dt^α ÷ α
    for i ← 1 to n do
        u_i ← A vector u ∈ ℝ^m depicting the deterministic control
        w_i ← A vector w ∈ ℝ^l depicting the process noise
        v_i ← A vector v ∈ ℝ^p depicting the measurement noise
        x_{i+1} ← x_i + factor × ξ(i, x_i)
        y_i ← C x_i + v_i
        K_i ← P_i C^T R^{-1}
        x̂_{i+1} ← x̂_i + factor × ξ(i, x̂_i)
        P_{i+1} ← P_i + factor × ξ(i, P_i)
    end for
end procedure
```

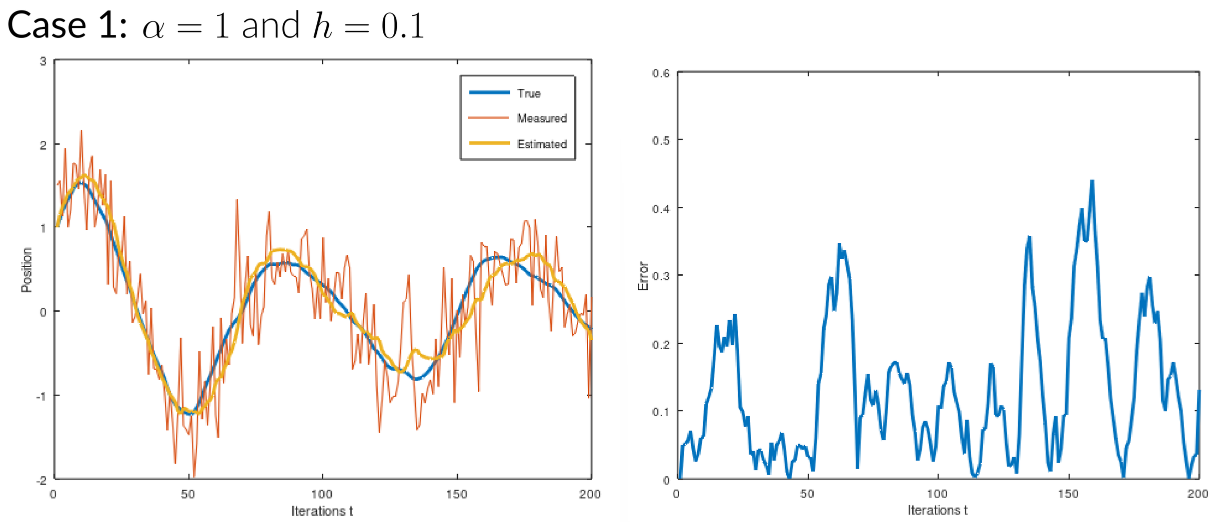
Linear Time Invariant (LTI)

Consider the stochastic oscillator

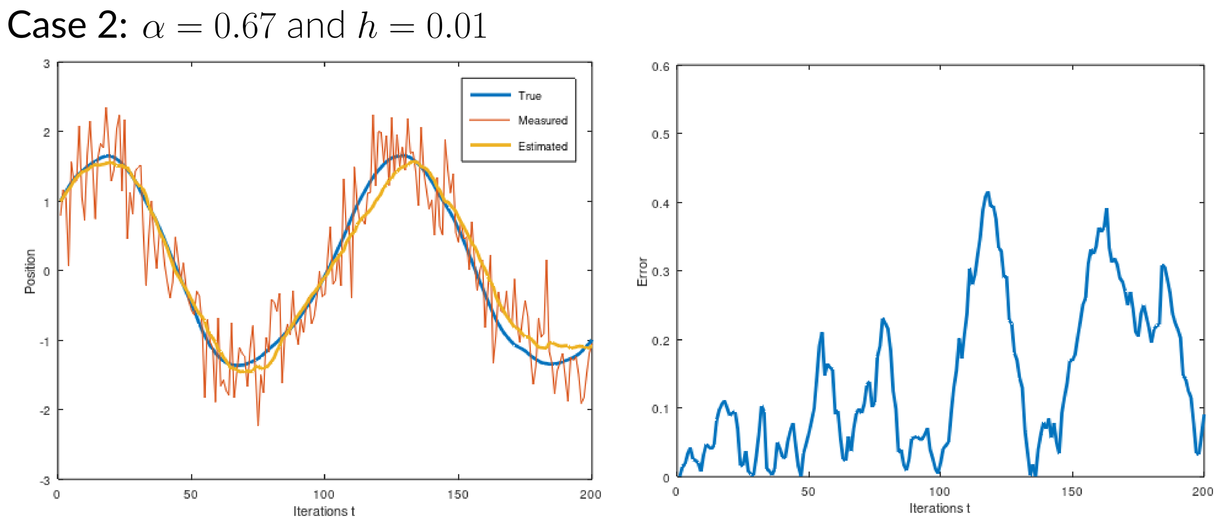
$$x^{(\alpha)}(t)=\begin{bmatrix}0 & 1\\ -0.64 & -0.25\end{bmatrix}x(t)+\begin{bmatrix}0\\1\end{bmatrix}w(t),\quad x(0)=\begin{bmatrix}1\\1\end{bmatrix}$$
$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}x(t)+v(t)$$

where $Q=3$, $R=2$, and $P(0)=\begin{bmatrix}2&0\\0&3\end{bmatrix}$. Here, we use $n=200$ iterations.

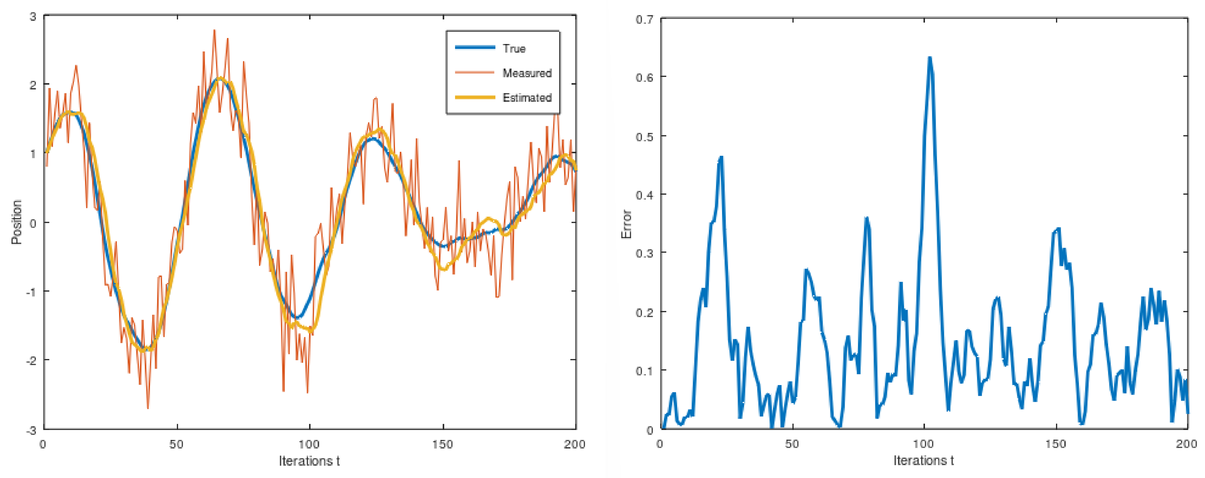
Case 1: $\alpha=1$ and $h=0.1$



Case 2: $\alpha=0.67$ and $h=0.01$



Case 3: $\alpha=0.34$ and $h=0.00001$



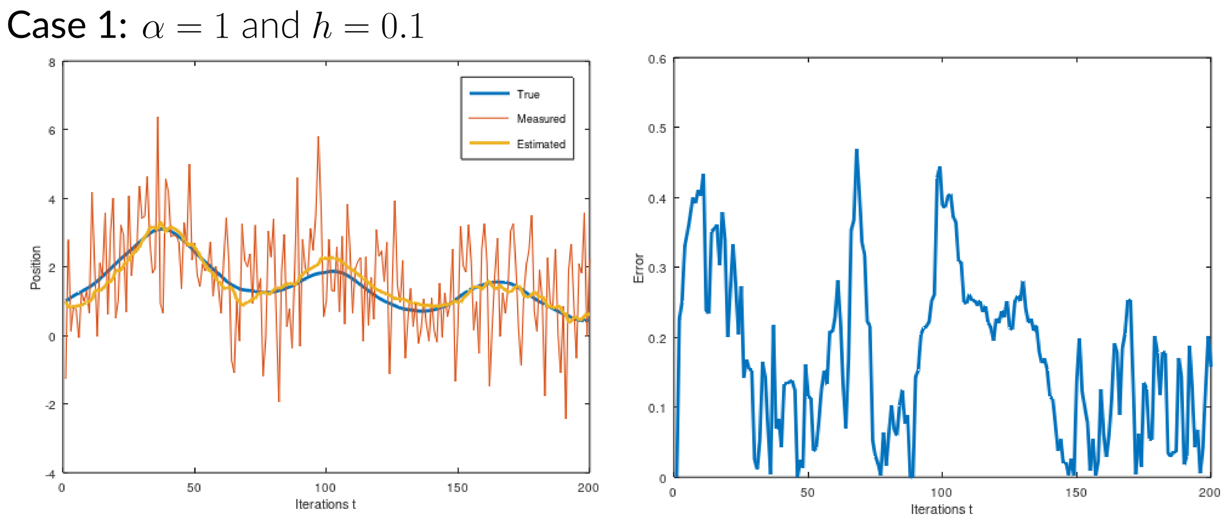
Linear Time Variant (LTV)

Consider the stochastic system

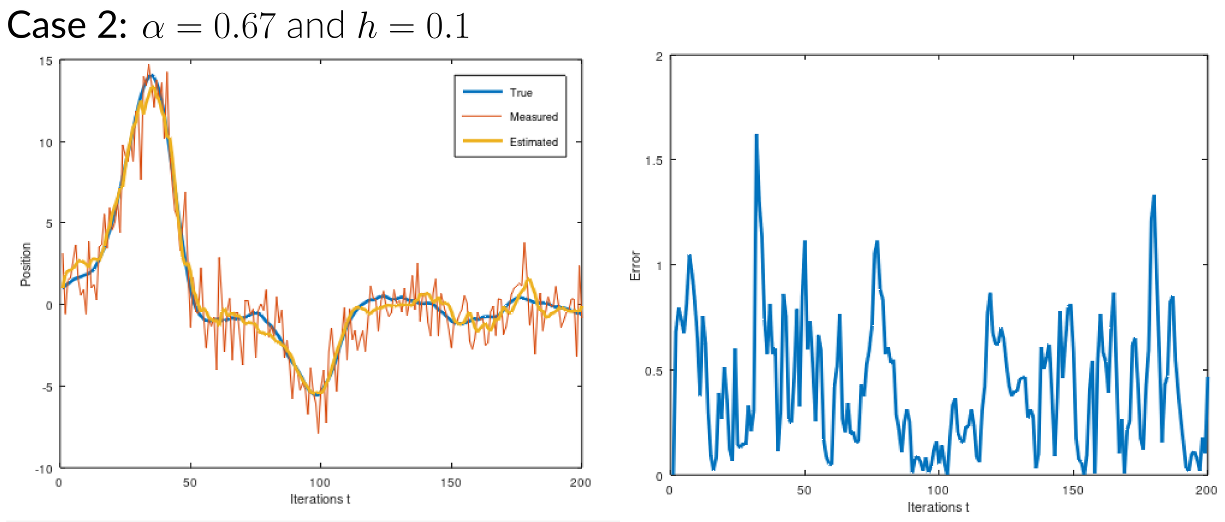
$$x^{(\alpha)}(t)=\begin{bmatrix}0 & 0.5\\ \sin(0.1t) & -1\end{bmatrix}x(t)+\begin{bmatrix}0\\1\end{bmatrix}w(t),\quad x(0)=\begin{bmatrix}1\\1\end{bmatrix}$$
$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}x(t)+v(t)$$

where $Q=3$, $R=2$, and $P(0)=\begin{bmatrix}2&0\\0&3\end{bmatrix}$. Here, we use $n=200$ iterations.

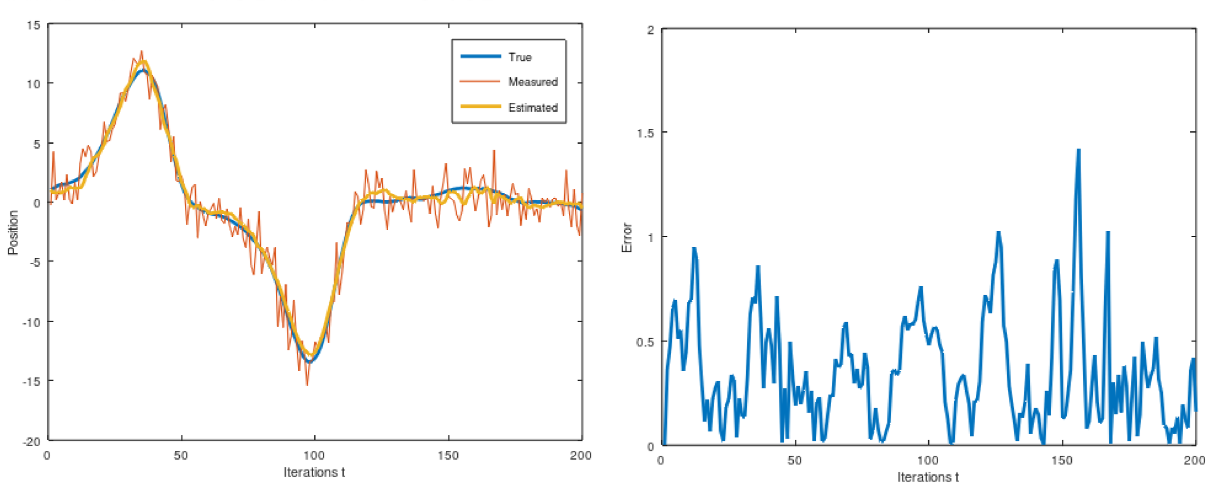
Case 1: $\alpha=1$ and $h=0.1$



Case 2: $\alpha=0.67$ and $h=0.1$



Case 3: $\alpha=0.34$ and $h=0.001$



Future Plans

Related work includes:

- Steady-state results
- Stability results through exponential weighting based on α
- Gain scheduling based on α
- Filter design with correlated noise
- Conformable information filter and corresponding smoother
- An extended conformable Kalman filter

References

[1] F.L. Lewis, L. Xie, and D. Popa. *Optimal and Robust Estimation: With an Introduction to Stochastic Control Theory.* CRC Press, 2008.

[2] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh. A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, 264:65–70, July 2014.

[3] A. Younus, T. Abdeljawad, and T. Gul. On stability criteria of fractal differential systems of conformable type. *Fractals*, 28(08):2040009, July 2020.

[4] T. Cuchta, D. Poulsen, and N. Wintz. Linear quadratic tracking with continuous conformable derivatives. *European Journal of Control*, 72:100808, July 2023.

Supported by NSF Grant DMS-2150226